Exact summation of the Chapman-Enskog expansion from moment equations

2000 J. Phys. A: Math. Gen. 33 8037

(http://iopscience.iop.org/0305-4470/33/45/303)

J. Phys. A: Math. Gen. **33** (2000) 8037–8046. Printed in the UK PII: S0305-4470(00)09429-4

Exact summation of the Chapman–Enskog expansion from moment equations

Iliya V Karlin

ETH-Zürich, Department of Materials, Institute of Polymer Physics, Sonneggstrasse 3, ML J 27, CH-8092 Zürich, Switzerland

E-mail: ikarlin@ifp.mat.ethz.ch

Received 9 November 1999, in final form 23 March 2000

Abstract. The nonlinear recurrence Chapman–Enskog solution for the linearized Grad ten-moment equations is resummed exactly. Using this solution, the stability of higher-order hydrodynamics in various approximations is discussed.

1. Introduction

The derivation of hydrodynamics from a microscopic description is the classical problem of physical kinetics. The Chapman–Enskog method [1] derives the solution from the Boltzmann equation in a form of a series in powers of the Knudsen number ϵ, where ϵ is the ratio between the mean free path of a particle and the scale of variations of hydrodynamic fields. The Chapman–Enskog solution leads to a formal expansion of the stress tensor and of the heat flux vector in balance equations for density, momentum and energy. Retaining the first-order term (ϵ) in the latter expansions, we come to the Navier–Stokes equations, while further corrections are known as the Burnett (ϵ^2) and the super-Burnett (ϵ^3) corrections [1].

However, as was first demonstrated by Bobylev [2], even in the simplest case (one-dimensional linear deviation from global equilibrium), the Burnett and the super-Burnett hydrodynamics violate the basic physics behind the Boltzmann equation. Namely, sufficiently short acoustic waves are increasing with time instead of decaying. This contradicts the H-theorem, since all near-equilibrium perturbations must decay. This creates difficulties for an extension of hydrodynamics, as derived from a microscopic description, into a highly non-equilibrium domain where the Navier–Stokes approximation is inapplicable. The latter problem remains one of the central open problems of kinetic theory (for the most recent contributions see [3, 4]).

In this paper, the problem of higher-order hydrodynamics is studied within the framework of exact solutions to simplified models. The linearized ten-moment Grad equations [5] are considered. The Chapman–Enskog method, as applied to this model, leads to a nonlinear recurrence solution which also suffers instabilities in the higher-order approximations. The result of the present study is the exact summation of the Chapman–Enskog expansion valid to arbitrary order in Knudsen number, and it extends the result [6] where the one-dimensional case has been considered. This result leads to a quantitative discussion of the Chapman–Enskog solution in the short-wave domain in frames of the model, and, in particular, it provides a test of various approximations used to extend the hydrodynamics.

0305-4470/00/458037+10$30.00

An additional motivation for this work is due to a recently introduced approach to subgrid turbulence modelling [7]. This approach makes use of hyperbolic supersets of the Navier–Stokes equations, such as the lattice Boltzmann equation or the discrete velocity models rather than the Navier–Stokes equation itself. One point of this approach is directly relevant to our study: the coarse-graining of the kinetic equations [7] allows us to consider higher-order hydrodynamics as a source for an effective flow-dependent viscosity. Since the lattice Boltzmann method is effectively a Grad-like system [8], our study is a relevant first step in this direction.

2. Summation of the Chapman–Enskog expansion

Throughout the paper, p and $\boldsymbol{u}$ are dimensionless deviations of the pressure and of the average velocity from their equilibrium values (see [9] for relations of these variables to dimensional quantities). The point of departure is the set of linearized Grad equations [9] for p, $\boldsymbol{u}$ and $\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ is the dimensionless stress tensor:

$$\begin{aligned}\partial_t p &= -\tfrac{5}{3}\nabla\cdot\boldsymbol{u}\\ \partial_t\boldsymbol{u} &= -\nabla p - \nabla\cdot\boldsymbol{\sigma}\\ \partial_t\boldsymbol{\sigma} &= -\overline{\nabla\boldsymbol{u}} - \frac{1}{\epsilon}\boldsymbol{\sigma}.\end{aligned} \tag{1}$$

The over-line will always denote a symmetric traceless dyad, thus

$$\overline{\nabla\boldsymbol{u}} = \nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\dagger} - \tfrac{2}{3}\mathsf{I}\nabla\cdot\boldsymbol{u}$$

where I is the unity matrix, and the dot denotes the standard scalar product. Equations (1) provides a simple model of a coupling of the hydrodynamic variables, $\boldsymbol{u}$ and p, to the non-hydrodynamic variable $\boldsymbol{\sigma}$, and corresponds to the case without heat conduction. These equations are suitable for an application of the Chapman–Enskog procedure. Therefore, our goal here is not to investigate the properties of equations (1) as they are, but to reduce the description to a closed set of equations with respect to the variables p and $\boldsymbol{u}$. That is, we have to express the tensor $\boldsymbol{\sigma}$ in terms of spatial derivatives of the hydrodynamic fields p and $\boldsymbol{u}$. The Chapman–Enskog method, as applied to equations (1) results in the following:

$$\boldsymbol{\sigma}^{\mathrm{CE}} = \sum_{n=0}^{\infty}\epsilon^{n+1}\boldsymbol{\sigma}^{(n)}. \tag{2}$$

Coefficients $\boldsymbol{\sigma}^{(n)}$ are due to the following recurrence procedure [9]:

$$\boldsymbol{\sigma}^{(n)} = -\sum_{m=0}^{n-1}\partial_t^{(m)}\boldsymbol{\sigma}^{(n-1-m)} \tag{3}$$

where the Chapman–Enskog operators $\partial_t^{(m)}$ act on p and on $\boldsymbol{u}$ as follows:

$$\begin{aligned}\partial_t^{(m)}\boldsymbol{u} &= \begin{cases}-\nabla p & m=0\\ -\nabla\cdot\boldsymbol{\sigma}^{(m-1)} & m\geqslant 1\end{cases}\\ \partial_t^{(m)}p &= \begin{cases}-\tfrac{5}{3}\nabla\cdot\boldsymbol{u} & m=0\\ 0 & m\geqslant 1.\end{cases}\end{aligned} \tag{4}$$

Finally, the zeroth-order term, $\boldsymbol{\sigma}^{(0)} = -\overline{\nabla\boldsymbol{u}}$, leads to the linearized Navier–Stokes hydrodynamics.

Because of a somewhat involved structure of the recurrence procedure (3) and (4), the Chapman–Enskog method is a nonlinear operation even in the model (1). Moreover, as was shown in [9], the instability is present: the acoustic mode in the Navier–Stokes and in the Burnett approximations are stable, while it is unstable in the super-Burnett approximation for sufficiently short waves.

Our goal is to sum up the series (2) in a closed form. First, we note that functions $\boldsymbol{\sigma}^{(n)}$ in equations (2)–(4), have the following explicit structure for arbitrary $n \geqslant 0$ [9]:

$$\begin{aligned}\boldsymbol{\sigma}^{(2n)} &= a_n \Delta^n \overline{\nabla \boldsymbol{u}} + b_n \Delta^{n-1} \mathsf{G} \nabla \cdot \boldsymbol{u} \\ \boldsymbol{\sigma}^{(2n+1)} &= c_n \Delta^n \mathsf{G} p\end{aligned} \tag{5}$$

where Δ is the Laplacian and $\mathsf{G} = \nabla\nabla - \frac{1}{3}\mathsf{I}\Delta$, while real-valued and yet unknown coefficients a_n, b_n and c_n are due to the recurrence procedure (3) and (4). Knowing the structure (5), it is not difficult to reformulate the Chapman–Enskog solution in terms of a recurrence procedure for the coefficients a_n, b_n and c_n. It is most convenient to make the Fourier transform. Taking $\boldsymbol{u} = \boldsymbol{u}_k \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x})$ and $p = p_k \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x})$ in (5), using (5) in equations (3) and (4), and after some algebra, we arrive at the following result:

$$\begin{aligned}c_{n+1} &= 2a_{n+1} + b_{n+1} + \tfrac{2}{3}\sum_{m=0}^{n}(2a_{n-m} + b_{n-m})c_m \\ a_{n+1}\overline{\boldsymbol{k}\boldsymbol{u}_k} + b_{n+1}\mathsf{g}_k(\boldsymbol{k}\cdot\boldsymbol{u}_k) &= \left(\sum_{m=0}^{n} a_{n-m}a_m\right)\overline{\boldsymbol{k}\boldsymbol{u}_k} \\ &\quad + \left(\tfrac{5}{3}c_n + \sum_{m=0}^{n}\left\{\tfrac{1}{3}(2a_{n-m} + b_{n-m})(a_m + 2b_m) + a_{n-m}b_m\right\}\right)\mathsf{g}_k(\boldsymbol{k}\cdot\boldsymbol{u}_k).\end{aligned} \tag{6}$$

Here $\mathsf{g}_k = \frac{1}{2}\overline{\boldsymbol{e}_k\boldsymbol{e}_k}$, and $\boldsymbol{e}_k$ is the unity vector directed along $\boldsymbol{k}$. The second of the equations in (6) is equivalent to two scalar equations. Introducing $r_n = \frac{2}{3}c_n$ and $q_n = \frac{2}{3}(2a_n + b_n)$, and using the identity, $\overline{\boldsymbol{k}\boldsymbol{u}_k} = (\overline{\boldsymbol{k}\boldsymbol{u}_k} - 2\mathsf{g}_k(\boldsymbol{k}\cdot\boldsymbol{u}_k)) + 2\mathsf{g}_k(\boldsymbol{k}\cdot\boldsymbol{u}_k)$, and also noticing that

$$\mathsf{g}_k : (\overline{\boldsymbol{k}\boldsymbol{u}_k} - 2\mathsf{g}_k(\boldsymbol{k}\cdot\boldsymbol{u}_k)) = 0$$

we arrive at the following three scalar recurrence relations:

$$\begin{aligned}r_{n+1} &= q_{n+1} + \sum_{m=0}^{n} q_{n-m}r_m \\ q_{n+1} &= \tfrac{5}{3}r_n + \sum_{m=0}^{n} q_{n-m}q_m \\ a_{n+1} &= \sum_{m=0}^{n} a_{n-m}a_m.\end{aligned} \tag{7}$$

The initial condition for this system reads: $r_0 = -\frac{4}{3}$, $q_0 = -\frac{4}{3}$, $a_0 = -1$.

The recurrence relations (7) are completely equivalent to the original Chapman–Enskog procedure (3) and (4). In the one-dimensional case, the recurrence system (7) reduces to the first two equations for r_n and q_n.

Now we will express the Chapman–Enskog series of the stress tensor (2) in terms of coefficients r_n, q_n and a_n. Using the Fourier transform again, and substituting equation (5) into equation (2), we derive

$$\boldsymbol{\sigma}_k^{\mathrm{CE}} = A(k^2)(\overline{\boldsymbol{k}\boldsymbol{u}_k} - 2\mathsf{g}_k(\boldsymbol{k}\cdot\boldsymbol{u}_k)) + \tfrac{3}{2}Q(k^2)\mathsf{g}_k(\boldsymbol{k}\cdot\boldsymbol{u}_k) - \tfrac{3}{2}k^2R(k^2)\mathsf{g}_k p_k. \tag{8}$$

From here on, we use a new spatial scale which amounts to $\boldsymbol{k}' = \epsilon\boldsymbol{k}$, and drop the prime. Functions $A(k^2)$, $Q(k^2)$ and $R(k^2)$ in equation (8) are defined by power series with coefficients (7):

$$A(k^2) = \sum_{n=0}^{\infty} a_n(-k^2)^n \qquad Q(k^2) = \sum_{n=0}^{\infty} q_n(-k^2)^n \qquad R(k^2) = \sum_{n=0}^{\infty} r_n(-k^2)^n. \tag{9}$$

Thus, the problem of summation of the Chapman–Enskog series (2) amounts to finding three functions, $A(k^2)$, $Q(k^2)$ and $R(k^2)$ (9) in the three- and two-dimensional cases, or to the two functions, $Q(k^2)$ and $R(k^2)$, in the one-dimensional case.

Now we will focus our attention on the problem of a computation of functions (9) from the recurrence equations (7). At this point, it is worthwhile to note that usual routes of dealing with the recurrence system (7) would be either to truncate it at a certain n, or to calculate all the coefficients explicitly, and next to substitute the result into the power series (9). Both these routes are not successful here. Indeed, already in the one-dimensional case, retaining the coefficients q_0, r_0 and q_1 gives the super-Burnett approximation which has a short-wave instability for $k^2 > 3$ [9], and there is no guarantee that the same will not occur in a higher-order truncation. On the other hand, a term-by-term computation of the whole set of coefficients is a quite non-trivial task because equations (7) are nonlinear.

Fortunately, another route is possible. Multiplying each of the equations in (7) by $(-k^2)^{n+1}$, and performing summation in n from zero to infinity, we obtain

$$\begin{aligned} Q - q_0 &= -k^2 \left\{ \tfrac{5}{3} R + \sum_{n=0}^{\infty} \sum_{m=0}^{n} q_{n-m} (-k^2)^{n-m} q_m (-k^2)^m \right\} \\ R - r_0 &= Q - q_0 - k^2 \sum_{n=0}^{\infty} \sum_{m=0}^{n} q_{n-m} (-k^2)^{n-m} r_m (-k^2)^m \\ A - a_0 &= -k^2 \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_{n-m} (-k^2)^{n-m} a_m (-k^2)^m . \end{aligned} \tag{10}$$

Now we note that (the Cauchy rule),

$$\begin{aligned} &\lim_{N\to\infty} \sum_{n=0}^{N} \sum_{m=0}^{n} a_{n-m} (-k^2)^{n-m} a_m (-k^2)^m = A^2 \\ &\lim_{N\to\infty} \sum_{n=0}^{N} \sum_{m=0}^{n} q_{n-m} (-k^2)^{n-m} r_m (-k^2)^m = QR \\ &\lim_{N\to\infty} \sum_{n=0}^{N} \sum_{m=0}^{n} q_{n-m} (-k^2)^{n-m} q_m (-k^2)^m = Q^2 . \end{aligned} \tag{11}$$

Taking into account the initial data, $q_0 = r_0 = -\frac{4}{3}$, $a_0 = -1$, and also using expressions (11), we come in equation (10) to the following three quadratic equations for functions A, R and Q:

$$\begin{aligned} Q &= -\tfrac{4}{3} - k^2 \left(\tfrac{5}{3} R + Q^2 \right) \\ R &= Q(1 - k^2 R) \\ A &= -(1 + k^2 A^2). \end{aligned} \tag{12}$$

The result (12) solves essentially the problem of computation of functions (9). Some further simplifications are possible. Introducing a new function, $X(k^2) = k^2 R(k^2)$, we derive a cubic equation:

$$-\tfrac{5}{3}(X-1)^2\left(X + \tfrac{4}{5}\right) = \frac{X}{k^2}. \tag{13}$$

We shall also rewrite the third equation in the system (12) using a function $Y(k^2) = k^2 A(k^2)$:

$$Y(1+Y) = -k^2. \tag{14}$$

Functions of interest (9) can be expressed in terms of relevant solutions to equations (13) and (14). Since all functions in equations (9) are real-valued, we are interested only in real-valued roots of equations (13) and (14).

Equation (13) was already discussed in [6]: the real-valued root $X(k^2)$ is unique and negative for all finite values of parameter k^2. This root is relevant to the extended acoustic mode. Moreover, function $X(k^2)$ is a monotonic function of k^2. Limiting values are

$$\lim_{k\to 0} X(k^2) = 0 \qquad \lim_{k\to\infty} X(k^2) = -\tfrac{4}{5}. \tag{15}$$

The quadratic equation (14) has no real-valued solutions for $k^2 > \frac{1}{4}$, and has two real-valued solutions for each k^2, where $k^2 < \frac{1}{4}$. We denote by $k_c = \frac{1}{2}$ the corresponding critical value of the wavevector. For $k = 0$, one of these roots is equal to zero, while the other is equal to one. The asymptotic $Y \to 0$, as $k \to 0$, answers the question of which these two roots is relevant to the Chapman–Enskog solution, and we derive

$$Y = \begin{cases} -\frac{1}{2}\left(1 - \sqrt{1-4k^2}\right) & k < k_c \\ \text{none} & k > k_c. \end{cases} \tag{16}$$

The function Y (16) is negative for $k \leqslant k_c$.

3. Hydrodynamic modes

The Fourier image of the function $\nabla \cdot \boldsymbol{\sigma}^{\mathrm{CE}}$ follows from equation (8):

$$\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{\sigma}_{\boldsymbol{k}}^{\mathrm{CE}} = Y((\boldsymbol{e_k} \cdot \boldsymbol{u_k})\boldsymbol{e_k} - \boldsymbol{u_k}) - \frac{X}{1-X}(\boldsymbol{e_k} \cdot \boldsymbol{u_k})\boldsymbol{e_k} - \mathrm{i}X\boldsymbol{k}p_{\boldsymbol{k}}. \tag{17}$$

This expression contributes to the right-hand side of the Fourier-transformed momentum equation (the second line in Grad's system (1)). Knowing (17), we calculate the dispersion relation $\omega(\boldsymbol{k})$ of plane waves $\sim \exp\{\omega t + \mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x}\}$. This computation is standard, and we only reproduce the final result. The exact dispersion relation of the hydrodynamic spectrum reads:

$$(\omega - Y)^{d-1}\left(\omega^2 - \frac{X}{1-X}\omega + \frac{5}{3}k^2(1-X)\right) = 0 \tag{18}$$

where d is the spatial dimension. From the dispersion relation (18), we easily derive the following classification of the hydrodynamic modes.

(a) For $d = 1$, the spectrum is purely acoustic with the dispersion ω_{a}:

$$\omega_a = \frac{X}{2(1-X)} \pm \mathrm{i}\frac{k}{2}\sqrt{\frac{5X^2 - 16X + 20}{3}} \tag{19}$$

where $X = X(k^2)$ is the real-valued root of equation (13). Since X is a negative function for all $k > 0$, the damping rate of acoustic modes, $\mathrm{Re}\,\omega_a$, is negative for all $k > 0$, and the exact acoustic spectrum of the Chapman–Enskog procedure is free of the instability for arbitrary wavelengths. In the short-wave limit, expression for the acoustic branch (19) reads

$$\lim_{k\to\infty} \omega_a = -\tfrac{2}{9} \pm \mathrm{i}k\sqrt{3}. \tag{20}$$

(b) For $d > 1$, the acoustic branch of the spectrum is given by equation (19). As follows from the Chapman–Enskog procedure, the shear mode has the dispersion ω_{d}:

$$\omega_{\mathrm{d}} = \begin{cases} -\frac{1}{2}\left(1 - \sqrt{1-4k^2}\right) & k < k_c \\ \text{none} & k > k_c. \end{cases} \tag{21}$$

The shear mode is $(d-1)$ times degenerated, the corresponding attenuation rate is negative for $k < k_c$, and this mode *cannot* be extended beyond the critical value $k_c = \frac{1}{2}$ within the Chapman–Enskog method.

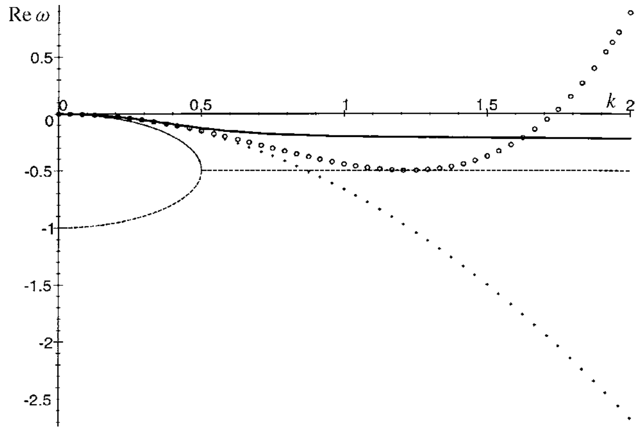

Figure 1. Attenuation rates (real parts of ω) as functions of the reduced wavevector k. Dots, the Navier–Stokes approximation, acoustic branch. Circles, the super-Burnett approximation, acoustic branch. The instability occurs when this line crosses the horizontal axis at $k = \sqrt{3}$. Bold curve, the exact Chapman–Enskog solution, acoustic branch (19). Full curve, the exact Chapman–Enskog solution, diffusion branch (21). Broken curve, the non-hydrodynamic branch of Grad's equations (1).

The reason why this singularity occurs can be found upon a closer investigation of the spectrum of the underlying Grad moment system (1). In the original system, there exist several non-hydrodynamic modes which are irrelevant to the Chapman–Enskog solution. All of these non-hydrodynamic modes are characterized by the property that corresponding dispersion relations $\omega(\boldsymbol{k})$ are not equal to zero at $k = 0$. In the point $k_c = \frac{1}{2}$, the branch (21) intersects with one of the non-hydrodynamic branches of equation (1). For larger k, these two branches produce a pair of complex-conjugated solutions with the real part equal to $-\frac{1}{2}$. Though the spectrum of Grad's equations (1) indeed continues after k_c, the Chapman–Enskog method *does not recognize this extension as a part of the hydrodynamic mode*.

Functions $\mathrm{Re}\,\omega_{\mathrm{a}}$ and $\mathrm{Re}\,\omega_{\mathrm{d}}$ are shown in figure 1, together with some approximations of the Chapman–Enskog method. The non-hydrodynamic branch of equations (1) which causes the breakdown of the Chapman–Enskog solution is also represented in figure 1. It is remarkable that, while the exact hydrodynamic description becomes inapplicable for the diffusion mode at $k \geqslant k_c$, the Navier–Stokes description is still providing a good approximation to the acoustic mode around this point. Finally, we remind the reader that all the results of this section are represented in reduced units: while the parameter ϵ enters the Chapman–Enskog procedure as a smallness parameter, this 'smallness' is meaningless in the sum of the Chapman–Enskog expansion. Formally, one can put $\epsilon = 1$ after the summation.

4. The stationary limit

It is instructive to discuss the stationary limit of the exact hydrodynamics. This point is non-trivial, in general, because the Chapman–Enskog method is applicable solely to time-dependent equations, and its stationary limit must be considered only after the solution to the first problem is obtained. Let us recall that for the linearized Boltzmann equation, the stationary limit of the Chapman–Enskog solution was first found by Grad [10] and later demonstrated in detail by Galkin [11]. The essence of this result is as follows: the exact linearized stationary hydrodynamics is represented by the linearized stationary Navier–Stokes equations, while the stationary Chapman–Enskog expansions of the stress tensor and of the heat flux degenerate to polynomials. This result is also valid for the linearized 13-moment Grad equations [12]. Let us discuss this point in some detail for our case.

For this purpose, let us come back to the Chapman–Enskog coefficients (5). First, we remark that in the stationary case the functions $\boldsymbol{u}$ and p in these expressions become solutions to equations with specified boundary conditions rather than the unspecified parameters they were in the non-stationary case. With this preliminary remark, let us represent the stationary Chapman–Enskog stress tensor as

$$\boldsymbol{\sigma}_{\mathrm{st}}^{\mathrm{CE}} = \boldsymbol{\sigma}_{\mathrm{st}}^{(0)} + \delta\boldsymbol{\sigma}_{\mathrm{st}}^{\mathrm{CE}} \tag{22}$$

where $\boldsymbol{\sigma}_{\mathrm{st}}^{(0)}$ is the stationary Navier–Stokes stress tensor. Substituting expression (22) into the stationary momentum equation (1), we have:

$$\begin{aligned} &\nabla \cdot \boldsymbol{u} = 0 \\ &\nabla p = -\nabla \cdot \boldsymbol{\sigma}_{\mathrm{st}}^{(0)} - \nabla \cdot \delta\boldsymbol{\sigma}_{\mathrm{st}}^{\mathrm{CE}}. \end{aligned} \tag{23}$$

Let us now *assume* that

$$\nabla \cdot \delta\boldsymbol{\sigma}_{\mathrm{st}}^{\mathrm{CE}} = 0 \tag{24}$$

on solutions to the stationary linearized Navier–Stokes equations which follow from equations (23) under the assumption (24):

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \Delta p = 0 \qquad \nabla p = \epsilon \Delta \boldsymbol{u}. \tag{25}$$

Furthermore, let us assume usual boundary conditions under which solutions to equations (25) are well defined (we do not need further details on the boundary conditions to complete the analysis). Now, from the structure of the Chapman–Enskog coefficients (5), it follows that all terms in $\delta\boldsymbol{\sigma}_{\mathrm{st}}^{\mathrm{CE}}$ become equal to zero on solutions to the Navier–Stokes equations (25), except for the functions $\boldsymbol{\sigma}_{\mathrm{st}}^{(1)}$ and $\boldsymbol{\sigma}_{\mathrm{st}}^{(2)}$, so that

$$\delta\boldsymbol{\sigma}_{\mathrm{st}}^{\mathrm{CE}} = \epsilon^2 c_0 \nabla\nabla p + \epsilon^3 a_1 \Delta(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\dagger}). \tag{26}$$

We stress that in this expression, functions $\boldsymbol{u}$ and p are solutions to equations (25), and cancellation means that all *equations* $\boldsymbol{\sigma}^{(n)} = 0$, $n \geqslant 3$, have solutions which also satisfy equation (25). Finally, it is straightforward to check that expression (26) verifies equation (24). This validates our assumption (24), and thus we have the following result: exact stationary hydrodynamic equations are given by the Navier–Stokes equations (25), while the exact stationary Chapman–Enskog stress tensor is a polynomial which contains the Navier–Stokes, the Burnett and the super-Burnett terms,

$$\boldsymbol{\sigma}_{\mathrm{st}}^{\mathrm{CE}} = -\epsilon(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\dagger}) + \epsilon^2 c_0 \nabla\nabla p + \epsilon^3 a_1 \Delta(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\dagger}). \tag{27}$$

This result fully agrees with the results for the linearized Boltzmann equation mentioned above. The analysis just presented is a sort of consistency between solutions to hydrodynamic equations and stress tensors. The final question is whether the limit just established is well defined, or, whether there are other representations,

$$\boldsymbol{\sigma}_{\mathrm{st}}^{\mathrm{CE}} = \boldsymbol{\sigma}'_{\mathrm{st}} + \boldsymbol{\sigma}''_{\mathrm{st}}$$

where $\boldsymbol{\sigma}'_{\mathrm{st}} \neq \boldsymbol{\sigma}_{\mathrm{st}}^{\mathrm{NS}}$, and where $\nabla \cdot \boldsymbol{\sigma}''_{\mathrm{st}} = 0$ on solutions to the equations,

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \nabla p = -\nabla \cdot \boldsymbol{\sigma}'_{\mathrm{st}}. \tag{28}$$

In order to answer this question, we note that under the usual boundary conditions, solutions to the Navier–Stokes equations (25) are also solutions to the equations just written. Thus, as soon as only standard boundary conditions are concerned, the stationary limit (27) is well defined. Non-standard boundary conditions, under which equations (28) may have solutions that do not reduce to the solutions of the Navier–Stokes equations, can, in principle, challenge this result (see a discussion in the next section).

It is remarkable that though the exact stationary linearized hydrodynamics is given by the Navier–Stokes equations, $\boldsymbol{\sigma}_{\mathrm{st}}^{\mathrm{CE}} \neq \boldsymbol{\sigma}_{\mathrm{st}}^{\mathrm{NS}}$. This remark is intended to prevent possible misconceptions: due to the simplicity of the system (1), it might seem that the stationary hydrodynamic equations (25) can be found directly, without using the Chapman–Enskog method. For instance, solving the stationary equation $\partial_t \boldsymbol{\sigma} = 0 = -\overline{\nabla \boldsymbol{u}} - \epsilon^{-1}\boldsymbol{\sigma}$, one obtains $\boldsymbol{\sigma} = -\epsilon\overline{\nabla \boldsymbol{u}}$. Substitution of the latter expression into the stationary momentum equation gives equations (25). Another possibility is as follows: eliminating $\boldsymbol{\sigma}$ right at the outset by means of differentiation in t, we arrive at the telegraph equation,

$$\partial_t^2 \boldsymbol{u} + \epsilon^{-1}\partial_t \boldsymbol{u} = \tfrac{5}{3}\nabla(\nabla \cdot \boldsymbol{u}) + \nabla \cdot \overline{\nabla \boldsymbol{u}} - \epsilon^{-1}\nabla p.$$

The stationary version again leads to the Navier–Stokes equations. There are certainly other possibilities to obtain the same result. However, along these lines one fails to derive the correct stationary Chapman–Enskog stress tensor (27), as well as to discuss the relevance of the boundary conditions. The reason is that none of these methods respects the correct order of the two operations: elimination of the time derivative (first), and taking the stationary limit (second). In the first step, the Chapman–Enskog method serves to define what is the stress tensor in terms of the hydrodynamic variables, while the second step uses the result of the first and the consistence treatment as demonstrated above. Thus, only the Chapman–Enskog method, or its analogues which deal with elimination of the time derivative, can be regarded as valid methods to derive hydrodynamics even for the stationary case. Also various time-independent conditions, such as the incompressibility condition, can be imposed only on after the Chapman–Enskog solution is established: if the incompressibility condition is imposed within equations (4), this affects the recurrence equations (6) and (7). Finally, the very fact that the Navier–Stokes equations appear in the true derivation and by addressing the stationary case directly is merely a coincidence caused by the simplicity of the system (1), and it is neither the case already for the linearized 13-moment equations nor for the nonlinear ten-moment equations.

5. Implications for modelling of extended hydrodynamics

The Grad equations like equation (1) can be viewed as the minimal kinetic model where the Chapman–Enskog method shows some of its features similar to the case of the Boltzmann equation. These features are: (a) the Chapman–Enskog method is the nonlinear recurrence procedure; (b) finite-order approximations suffer the instability; (c) there exists a crossover of the diffusion-like hydrodynamic mode to a non-hydrodynamic mode (similar behaviour is know for the kinetic Lorentz gas model). Because exact summation of the Chapman–Enskog expansion for the Boltzmann equation is a difficult unsolved problem even in the linearized case, and also because the Chapman–Enskog expansion for much simpler models like the one discussed in this paper have definite common points with the 'true' case, it is appealing to think of using them to suggest models of generalized higher-order hydrodynamics. We remark that while the linearized case is also solved for Grad's 13-moment approximation, where the heat flux is included [13], the situation is much more complicated for nonlinear Grad equations where only partial results have been obtained so far [14]. We indicate here how suggestions for extended hydrodynamics should be made.

The approximate generalized hydrodynamics is thought of as the result of approximations carried upon the Chapman–Enskog recurrence procedure as a whole, and its validity is tested by a comparison with the results of exact summation. To this end, let us impose the incompressibility condition, $\nabla \cdot \boldsymbol{u}(\boldsymbol{x}, t) = 0$, and let us also assume that the stationary pressure field satisfies the Laplace equation, $\Delta p(\boldsymbol{x}) = 0$. With this, the Chapman–Enskog solution reduces to the following recurrence relations (see equations (5) and (7)):

$$\boldsymbol{\sigma}^{(2n)} = a_n \Delta^n \overline{\nabla \boldsymbol{u}}$$
$$a_{n+1} = \sum_{m=0}^{n} a_{n-m} a_m \tag{29}$$
$$a_0 = -1.$$

Let us now approximate the nonlinear recurrence relation by the linear relation [9]:

$$a_{n+1} = a_0 a_n \qquad a_0 = -1. \tag{30}$$

Summation of the Chapman–Enskog expansion with coefficients (30) results in the stress tensor,

$$\boldsymbol{\sigma} = a_0 \epsilon (1 - a_0 \epsilon^2 \Delta)^{-1} \overline{\nabla \boldsymbol{u}} \tag{31}$$

while the non-stationary momentum equation reads,

$$\partial_t \boldsymbol{u}(\boldsymbol{x}, t) = -\nabla p(\boldsymbol{x}) - a_0 \epsilon (1 - a_0 \epsilon^2 \Delta)^{-1} \Delta \boldsymbol{u}(\boldsymbol{x}, t). \tag{32}$$

Approximations of the form (31) were suggested earlier [9, 15] as a way to regularize the instability of the higher-order approximations for the acoustic part of the extended hydrodynamics, systematic procedures to extend this approximation have also been discussed [9]. Here the approximation (32) is relevant to the diffusion-like part. As we have learned from the exact summation, the Chapman–Enskog solution does not exist after $k > k_c$. The approximation (31) reflects this feature by a pole at $k_p = |a_0|^{-1/2} \epsilon^{-1}$: the decay rate tends to $-\infty$ as $|k| \to k_p$ from the left, thus, no hydrodynamics is excited above this value. In the stationary limit, it can be demonstrated that the simplest non-Navier–Stokes equation associated with equation (32) is as follows:

$$(1 + a_0 \epsilon^2 \Delta) \Delta \boldsymbol{u}(\boldsymbol{x}) = \nabla p(\boldsymbol{x}). \tag{33}$$

The demonstration goes along the same lines as in the preceding section. If boundary conditions for equation (33) are such that the stationary Navier–Stokes equations have a solution then we return to the above discussion. However, equation (33) is of higher (fourth) order, and it also has solutions for boundary conditions which are invalid for the Navier–Stokes equation. The example discussed here reflects the following expectation concerning higher-order hydrodynamics: on the one hand, in the non-stationary case, the stress tensor should be represented by a highly non-local operator in order to be able to reproduce the asymptotics of the linear hydrodynamics as given by the Chapman–Enskog solution. On the other hand, in the stationary case, this non-local operator should reduce to a polynomial so that the boundary conditions can be discussed in a straightforward way. Finally, a comment is in order concerning the validity of the extended hydrodynamics. Though the Grad equations originate from the Boltzmann equation for the dilute gas, the structure of these equations is quite general, for instance, many of the well known constitutive equations in rheology have this type of coupling between the hydrodynamic and non-hydrodynamic degrees of freedom [16]. Thus, it is expected that the universality of the extended hydrodynamic equations is similar to the universality of the Navier–Stokes equations. It is not excluded that the non-trivial boundary conditions are the key to experimental verification of the extended hydrodynamics.

6. Conclusions

In this paper, we have found the sum of the Chapman–Enskog expansion for the minimal kinetic model (1). This result demonstrates the following.

(a) The acoustic-like mode of the exact hydrodynamics extends to all values of the wavevector and is stable. This extension contrasts sharply with its finite-order approximations which are unstable.
(b) The Chapman–Enskog solution for the diffusion-like mode extends only to the crossover value of the wavevector. After the crossover, the Chapman–Enskog solution does not exist. Any finite-order approximation, while stable, does not indicate the crossover and therefore is qualitatively wrong at large wavenumbers. These results demonstrate that taking account of the Chapman–Enskog expansion to all the orders is unavoidable to extend the non-stationary hydrodynamics beyond the Navier–Stokes approximation.
(c) The stationary limit of the exact Chapman–Enskog solution is shown to be consistent with the known result for the linearized Boltzmann equation. The importance of the boundary conditions and of the correct transition to the stationary limit have been stressed. We have also discussed how the minimal kinetic models can be of use for suggesting extended hydrodynamics.

Acknowledgments

It is my pleasure to thank Sauro Succi for his explanations of the paper [7], and Hans Christian Öttinger and Alexander Gorban for motivating discussions.

References

[1] Chapman S and Cowling T 1970 *Mathematical Theory of Non-Uniform Gases* 3rd edn (Cambridge: Cambridge University Press)
[2] Bobylev A V 1982 *Sov. Phys.–Dokl.* **27** 29
[3] Uribe F J and García-Colín L S 1999 *Phys. Rev.* E **60** 4052
[4] Uribe F J and Garcia A L 1999 *Phys. Rev.* E **60** 4063
[5] Grad H 1949 *Commun. Pure Appl. Math.* **2** 331
[6] Gorban A N and Karlin I V 1996 *Phys. Rev. Lett.* **77** 282
[7] Chen H, Succi S and Orszag S 1999 *Phys. Rev.* E **59** R2527
[8] Benzi R, Succi S and Vergassola M 1992 *Phys. Rep.* **222** 147
[9] Gorban A N and Karlin I V 1991 *Sov. Phys.–JETP* **73** 637
[10] Grad H 1963 *Phys. Fluids* **6** 147
[11] Galkin V 1988 *Mech. Jidkosti i Gaza* **4** 157
[12] Gorban A N and Karlin I V 1992 *Transp. Theory Stat. Phys.* **21** 101
[13] Karlin I V and Gorban A N 2000 Hydrodynamics from Grad's equations: what can we learn from exact solutions? submitted
[14] Karlin I V, Dukek G and Nonnenmacher T 1997 *Phys. Rev.* E **55** 1573
[15] Rosenau P 1989 *Phys. Rev.* A **40** 7193
[16] Bird R B, Armstrong R C and Hassager O 1987 *Dynamics of Polymeric Fluids, vol 1, Fluid Mechanics* 2nd edn (New York: Wiley)